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A Remark on the Hydrodynamics of the Zero-Range Processes

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The nonequilibrium stationary hydrodynamical properties of the symmetric nearest neighbor zero-range processes are studied: local equilibrium and Fourier's law are proven to hold, and the bulk diffusion coefficient and the equal time covariance of the limiting nonequilibrium stationary density fluctuations field are computed. The result fits with those already known and confirms some conjectures derived from a time-dependent macroscopic analysis. The very simple proof is based on a result already published but may be not so well known in this context.

KEY WORDS: Hydrodynamical behavior of microscopic systems; stochastic dynamics; zero range processes; local equilibrium; Fourier's law.

Systems of infinitely many, stochastic, mutually interacting particles may exhibit some of the hydrodynamical features of real systems. Their theory is by now fairly well established; cf. Ref. 3 and references cited therein. In this paper we consider the zero-range processes and we prove the validity of the nonequilibrium steady state hydrodynamical properties. The triviality of the proofs is in some sense unexpected but, as we will see, it is due to some factorization properties special to these models.

The zero-range model is the process described by $\{\eta(x,t),\ x\in\mathbb{Z},\ t\geqslant 0\}$ where $\eta(x,t)\in\mathbb{Z}^+$ denotes the number of particles at site x at time t. The process is a jump Markov process with state space $\mathbb{N}^{\mathbb{Z}}$; the dynamics is completely specified by the jump rates $g(\eta(x))$, $x\in\mathbb{Z}$, which give the rate for one of the $\eta(x)$ particles at site x to jump to one of the nearest-neighbor sites $x+1, x-1$ with probability $1/2$. The function $g(k)$, $k\in\mathbb{N}$ is non-

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decreasing, $g(0) = 0$, and

$$\sup_k (g(k+1) - g(k)) = c < +\infty$$

This process has been introduced by Spitzer.[11] In Ref. 1, for instance, the existence of the process and its ergodic properties have been studied. The extremal invariant measures are the product measures ν_p, $p \in [0, \sup_k g(k))$ given by (see Ref. 1) [$\mu(\cdot)$ denotes expectation with respect to μ]

$$\nu_p(\{\eta(0) = k\}) = \begin{cases} Z^{-1} \dfrac{p^k}{g(1) \ldots g(k)}, & k \neq 0 \\ Z^{-1}, & k = 0 \end{cases} \tag{1a}$$

$$Z = 1 + \sum_{k=1}^{\infty} \frac{p^k}{g(1) \cdots g(k)} \tag{1b}$$

$$p = \nu_p(g(\eta(0))) \tag{1c}$$

$$\rho = \nu_p(\eta(0)) \tag{1d}$$

The hydrodynamical equations for this model are conjectured to be

$$\frac{\partial \rho(q,\tau)}{\partial \tau} = \frac{\partial}{\partial q}\left[D(\rho(q,\tau)) \frac{\partial \rho}{\partial q} \right] \tag{2a}$$

where

$$D(\bar{\rho}) = -2 \left. \frac{dp}{d\rho} \right|_{\rho=\bar{\rho}} \tag{2b}$$

The conjecture is based on some heuristic arguments first employed by Morrey to derive the Euler equations from the microscopic equations of motion.[3,9,10] The approach has been extended to some stochastic evolutions.[3] Here we prove the validity of Eq. (2) in the nonequilibrium stationary case. This is a straightforward consequence of a result of Refs. 1 and 7; cf. Theorem 1 below. We think that the result deserves to be communicated even though its derivation is trivial. We have two motivations. The first one is that a rigorous derivation of the hydrodynamical equations is a hard problem, it has been accomplished just in a few models (see Ref. 3 and references quoted therein). Here we see, however, that for a class of systems, the zero-range symmetric processes nonequilibrium stationary hydrodynamics can be rigorously derived. [For such processes results are known for the particular case $g(k) = 1$ $\forall k \neq 0$,[5] where an extensive analysis of the hydrodynamical properties is available. The linearized hydrodynamical theory and the fluctuation–dissipation theorem have been recently proven by Brox and Rost[2] for the process we consider here.] Another reason which motivated this paper comes from a remark of Herbert Spohn. He showed us how to guess the equal-time nonequilibrium stationary covariance of the density fluctuation field. He then remarked that, owing to some surprising cancellations, in the zero-range process the resulting covariance is diagonal. Since one conjectures that there is a relation between macroscopic observables and the microscopic structure of the system, we thought that some analogous simplifications had to occur also in the nonequilibrium stationary process. We then realized what is now stated in Theorem 1. However before exposing our results, we think it might be useful to report (in our language) Spohn's argument, which we do very briefly in the following. Suppose $\{\mu_\epsilon\}_{\epsilon\in(0,1)}$ is a suitable family of measures which describe the equilibrium profile $\rho(q,\tau)$ given by Eq. (3); see Ref. 3. Define the nonequilibrium density fluctuation field as ($f \in s(\mathbb{R})$):

$$Y_\tau^\epsilon(f) = \epsilon^{1/2}\sum_x f(\epsilon x)\Big[\eta(x,\epsilon^{-2}\tau) - \mu^\epsilon\big(\eta(x,\epsilon^{-2}\tau)\big)\Big] \tag{3}$$

Then one expects [this has been actually proven in Ref. 5 for the case $g(k)=1$, $\forall k \neq 0$] that $\{Y_\tau^\epsilon(f)\}$ on $D([0,\infty), s'(\mathbb{R}))$ converges to a Gaussian, generalized Ornstein–Uhlenbeck process $\{Y_\tau(\cdot)\}$. $Y_\tau(f)$ is completely characterized, once the time zero distribution is given, by the fact that for each $\Phi \in C_0^\infty(\mathbb{R})$ its law $\mathscr{P}$ makes the following process a martingale[6]:

$$M_\tau(\Phi; f) = \Phi_\tau - \int_0^\tau d\tau'\, Y_{\tau'}(A_{\tau'} f)\Phi'_{\tau'} - \frac{1}{2}\|B_\tau f\|^2 \int_0^\tau d\tau'\, \Phi''_{\tau'} \tag{4a}$$

$$\Phi_\tau = \Phi(Y_\tau(f)), \qquad \Phi'_\tau = \left(\frac{d}{dx}\Phi\right)(Y_\tau(f)), \qquad \Phi''_\tau = \left(\frac{d^2}{dx^2}\Phi\right)(Y_\tau(f)) \tag{4b}$$

$$A_\tau f = \frac{1}{2} D(\rho(q,\tau))\frac{\partial^2 f}{\partial q^2} \tag{4c}$$

$$\|B_\tau f\|^2 = \int dq\, f'(q)^2 D(\rho(q,\tau))\chi(\rho(q,\tau)) \tag{4d}$$

$$\chi(\rho) = \text{compressibility} := \nu_{p(\rho)}\big((\eta(0)-\rho)^2\big) \tag{4e}$$

The equal time covariance, $C_\tau(q,q')$, is

$$\mathscr{P}(Y_\tau(f)Y_\tau(g)) = \int dq\, dq'\, f(q)C_\tau(q,q')g(q') \tag{5}$$

from Eq. (4) one finds

$$C_\tau = C_0 + \hat{C}_\tau, \qquad C_0(q,q') = \delta(q-q')\chi(\rho(q',0)) \tag{6a}$$

$$\frac{\partial \hat{C}_\tau}{\partial \tau} = A_\tau C_\tau + C_\tau A_\tau^* + \frac{1}{2}(D\chi)''\mathbb{1}, \qquad \hat{C}_0 = 0 \tag{6b}$$

$\mathbb{1}$ is the diagonal operator, $(D\chi)'' = \partial^2/\partial q^2(D\chi)$ (6c)

So, in the stationary case the covariance will be formally given by

$$C = C_0 + \frac{1}{2}\int_0^{\infty} d\sigma\, e^{-A\sigma}(D\chi)'' e^{-A^*\sigma} \tag{7}$$

Using Eq. (2) in the stationary case we have that

$$(D\chi)'' = \frac{d}{dq}\left(\frac{d}{d\rho}(D\chi)\frac{d\rho}{dq}\right) = \frac{d}{dq}\left(D\frac{\delta\rho}{dq}\right) = 0 \tag{8}$$

So in any zero-range model $C = C_0$. Here we used the identity Eq. (10) below (special to zero-range models):

$$\frac{d\rho}{dp} = \frac{1}{p}\chi(\rho) \tag{9}$$

for which [since $D = -2(dp/d\rho)$]

$$D\chi = -2\chi\frac{dp}{d\rho} = -2p \tag{10}$$

Let us consider the system in $[-L, L]$ and put boundary conditions r_-, r_+ at $\mp L$. The generator is Ω_L:

$$(\Omega_L f)(\eta) = \frac{1}{2}\sum_{x=-L}^{L} g(\eta(x))\left[f(\eta^{x,x+1}) + f(\eta^{x,x-1}) - 2f(\eta)\right]$$
$$+ \frac{1}{2}r_-\left[f(\eta^{-L}) - f(\eta)\right] + \frac{1}{2}r_+\left[f(\eta^{L}) - f(\eta)\right] \tag{11}$$

where

$$\eta^{x,y}(z) = \eta(z),\quad z \neq x, y,\quad \eta^{x,y}(x) = \eta(x) - 1,\quad \eta^{x,y}(y) = \eta(y) + 1,$$
$$x, y \neq \pm L$$

$$\eta^{L,L+1}(L) = \eta(L) - 1,\quad \eta^{-L,-L-1}(-L) = \eta(-L) - 1$$

and η^L (η^{-L}) is the configuration η adding a particle at the point L ($-L$).

Theorem 1. Let r_+, r_-, $r_+ \neq r_-$, be nonnegative real numbers such that

$$\max(r_+, r_-) \leqslant \sup_k g(k) \tag{12}$$

Then the unique invariant measure μ_L for the process with generator Ω_L is a product measure with marginal distributions:

$$\mu_L(\{\eta(x) = k\}) = \frac{p_L(x)^k}{g(1)\cdots g(k)} Z_L^{-1}(x),\qquad k \neq 0,\quad -L \leqslant x \leqslant L \tag{13a}$$

$$p_L(x) = \frac{r_+ - r_-}{2(L+1)}x + \frac{r_+ + r_-}{2} \tag{13b}$$

Proof. Can be found in Ref. 7 or by direct computation. We knew it from Ref. 1, Section 8. The condition (12) comes from the following fact. To define correctly the measure μ_L we need that

$$Z_L(x) = 1 + \sum_{k=1}^{\infty} \frac{p_L(x)^k}{g(1) \cdots g(k)} < \infty, \qquad -L \leqslant x \leqslant L \tag{14}$$

This is ensured by requiring

$$\max(p_L(L), p_L(-L)) < \sup_k g(k)$$

where [see Eq. (13b)]

$$p_L(\pm L) = \frac{L}{L+1} r_\pm + \frac{r_+ + r_-}{2(L+1)} \qquad \blacksquare$$

As a corollary we can derive the hydrodynamical equation (2) in the stationary case.

Theorem 2. Let r_-, r_+ be such that Eq. (12) holds, and let μ_L be the unique invariant measure for the Ω_L process. Then the following hold.

(i) *Local Equilibrium and L^{-1} Correction.* Let $q \in (-1, 1)$ and let $q_L = [qL]/(L+1)$, where $[a]$ denotes the integer part of a. Then for any cylindrical function f (τ_x denotes the shift by $x \in \mathbb{Z}$) we have

$$\lim_{L\to\infty} |\mu_L(\tau_{[qL]} f) - \nu_{p(q_L)}(f)| = 0 \tag{15a}$$

$$\lim_{L\to\infty} |\mu_L(\tau_{[qL]} f) - \nu_{p(q_L)}(f)| = \alpha(q) \sum_{x\in\mathbb{Z}} x\nu_{p(q)}((\eta(x) - \rho)f) \tag{15b}$$

where $\nu_{p(q)}$ is the equilibrium measure with parameter $p(q)$ and density $\rho(q)$:

$$p(q) = \tfrac{1}{2}(r_+ - r_-)q + \tfrac{1}{2}(r_+ + r_-) \tag{16a}$$

and $\rho(q)$ is the unique solution of

$$\frac{d}{dq}\left(D(\rho(q))\frac{d\rho}{dq}\right) = 0, \qquad -1 < q < 1$$
$$\rho(-1) = \rho_-, \qquad \rho(1) = \rho_+ \tag{16b}$$

$$\rho_\pm = \nu_{r_\pm}(\eta(0)) \tag{16c}$$

$$D(\rho(q)) = -2 \left.\frac{dp}{d\rho}\right|_{\rho=\rho(q)} \tag{16d}$$

$$\alpha(q) = \frac{1}{p(q)} \frac{1}{2}(r_+ - r_-) = \chi^{-1} \left.\frac{d\rho}{dq'}\right|_{q'=q} \tag{16e}$$

In the last equality of Eq. (16e) we have used Eq. (9). Equations (16) are a straightforward consequence of the product structure of the measure μ_L.

(ii) *Fourier's Law.* The current $j(x, x+1)$ at the bond $(x, x+1)$ is defined by

$$\Omega_L \eta(x) = -\left[j(x, x+1) - j(x-1, x) \right] \tag{17}$$

Then the Fourier's law holds, namely,

$$\lim_{L\to\infty} L\mu_L(j(x+[Lq], x+1+[Lq])) = -D(\rho(q)) \left. \frac{d\rho}{dq'} \right|_{q'=q} \tag{18}$$

Equation (18) is an immediate consequence of Eq. (15b): note that the average current at equilibrium is zero and that [by definition Eq. (17)]

$$j(x, x+1) = \tau_{x+1} g(\eta(0)) - \tau_x g(\eta(0))$$

(iii) *The Density Fluctuation Field.* The density fluctuation field is defined by

$$Y^L(f) = \frac{1}{\sqrt{L}} \sum_{x=-L}^{L} f\left(\frac{x}{L}\right)(\eta(x) - \mu_L(\eta(x))) \tag{19}$$

Then the limit covariance is given by

$$\lim_{L\to\infty} \mu_L\left(Y^L(f) Y^L(g)\right) = \int dq\, f(q)\chi(\rho(q))g(q) \tag{20}$$

Actually Theorem 1 holds for a process constructed on $\mathbb{N}^S$, S being a denumerable space, with sources on a set $T \subset S$.[1] One can then prove Theorem 2 in any dimension. In this case the symmetric n.n. process Ω_L would be constructed with sources in the hyperplanes $\{x_1 = \pm L\}$ (x_1 is the first coordinate of $x \in \mathbb{Z}^d$). The invariant measures will still be product measures such that $\mu_L(g(\eta(x))) = p_L(x_1)$ where p_L is defined in Eq. (13b).

Concluding Remarks. In some sense the triviality of the nonequilibrium stationary case is unexpected. The symmetric simple exclusion process is considered rather trivial because it is self-dual and so its analysis is easier than that of the other stochastic models. Furthermore the bulk diffusion coefficient is constant, $D = 1$. On the other hand for this model the L^{-1} correction and the nonequilibrium stationary covariance are not trivial. In fact, in Ref. 4 it has been proven that in the L^{-1} approximation correlations remain finite and they are equal to the nontrivial structure of the density fluctuation field; in Ref. 12 it is proven that the nonequilibrium stationary fluctuation field Y^L converges to a Gaussian process with covariance given by Eq. (7) [in this model $\chi = \rho - \rho^2$, $D = 1$, so $(D\chi)'' = -2(d\rho/dq)^2$].

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